

A DISCRETE BAKRY-EMERY METHOD AND ITS APPLICATION TO THE POROUS-MEDIUM EQUATION

ANSGAR JÜNGEL AND STEFAN SCHUCHNIGG

ABSTRACT. The exponential decay of the relative entropy associated to a fully discrete porous-medium equation in one space dimension is shown by means of a discrete Bakry-Emery approach. The first ingredient of the proof is an abstract discrete Bakry-Emery method, which states conditions on a sequence under which the exponential decay of the discrete entropy follows. The second ingredient is a new nonlinear summation-by-parts formula which is inspired by systematic integration by parts developed by Matthes and the first author. Numerical simulations illustrate the exponential decay of the entropy for various time and space step sizes.

1. INTRODUCTION

The Bakry-Emery method allows one to establish convex Sobolev inequalities and to compute exponential decay rates towards equilibrium for solutions to diffusion equations [2, 3]. The key idea of Bakry and Emery is to differentiate a so-called entropy functional twice with respect to time and to relate the second-order derivative to the entropy production. Our aim is to develop a discrete version of this technique, and in this paper, we present a step forward in this direction.

The study of discrete Bakry-Emery methods and related topics is rather recent. Caputo et al. [5] computed exponential decay rates for time-continuous Markov processes, using the Bochner-Bakry-Emery method. Given a stochastic process with density $u(t)$ and the entropy functional $H_c(u(t))$, the core of the Bakry-Emery approach is to find a constant $\lambda > 0$ such that the inequality $d^2 H_c/dt^2 \geq -\lambda dH_c/dt$ holds for all time. Integrating this inequality, one may show that $dH_c/dt \leq -\lambda H_c$ which implies that $H_c(u(t)) \leq e^{-\lambda t} H_c(u(0))$ for all $t > 0$, i.e., the entropy decays exponentially fast along $u(t)$. The relation between $d^2 H_c/dt^2$ and dH_c/dt is achieved in [5] by employing a discrete Bochner-type identity which replaces the Bochner identity of the continuous case. The Bochner-Bakry-Emery method was extended by Fathi and Maas in [13] in the context of Ricci curvature bounds and used by the authors of [19] to derive discrete Beckner inequalities.

Date: February 14, 2017.

2000 Mathematics Subject Classification. 65J08, 65M06, 65M12, 65Q10.

Key words and phrases. Finite differences, Bakry-Emery method, large-time asymptotics, systematic integration by parts.

The authors acknowledge partial support from the Austrian Science Fund (FWF), grants P22108, P24304, and W1245, and the Austrian-French Program of the Austrian Exchange Service (ÖAD).

Another approach has been suggested by Mielke [21]. He investigated geodesic convexity properties of nonlocal transportation distances on probability spaces such that continuous-time Markov chains can be formulated as gradient flows. Very related results have been obtained independently by Chow et al. [10] and Maas [20]. The geodesic convexity property implies exponential decay rates [1]. Mielke showed that the inequality $d^2 H_c/dt^2 \geq -\lambda dH_c/dt$ is equivalent to the positive semi-definiteness of a certain matrix such that matrix algebra can be applied. This idea was extended recently to certain nonlinear Fokker-Planck equations [7].

All these examples involve spatial semi-discretizations of diffusion equations. Temporal semi-discretizations often employ the implicit Euler scheme since it gives entropy dissipation, $dH_c/dt \leq 0$, under rather general conditions; see, e.g., the implicit Euler finite-volume approximations in [8, 14]. Entropy-dissipating higher-order semi-discretizations have been analyzed in [11, 17, 18]. However, there seem to be no results for fully discrete schemes using the Bakry-Emery approach. In this paper, we make a first step to fill this gap.

In order to understand the mathematical difficulty in fully discrete schemes, consider the abstract Cauchy problem

$$(1) \quad \partial_t u + A(u) = 0, \quad t > 0, \quad u(0) = u^0,$$

where $A : D(A) \rightarrow X'$ is a (nonlinear) operator defined on its domain $D(A) \subset X$ of the Banach space X with dual X' . If the dual product $\langle A(u), H'_c(u) \rangle$ is nonnegative, where $H'_c(u)$ is the (Fréchet) derivative of the entropy and $u(t)$ a solution to (1), then

$$\frac{dH_c}{dt} = \langle \partial_t u, H'_c(u) \rangle = -\langle A(u), H'_c(u) \rangle \leq 0,$$

showing entropy dissipation. Next, consider the implicit Euler scheme

$$\tau^{-1}(u^k - u^{k-1}) + A_h(u^k) = 0, \quad k \in \mathbb{N}, \quad \tau > 0,$$

where u^k is an approximation of $u(k\tau)$ and A_h is an approximation of A still satisfying $\langle A_h(u^k), H'(u^k) \rangle \geq 0$. Here, $H(u^k)$ is the discrete entropy, which is supposed to be convex. Then entropy dissipation is preserved by the scheme since

$$(2) \quad H(u^k) - H(u^{k-1}) \leq \langle u^k - u^{k-1}, H'(u^k) \rangle = -\tau \langle A_h(u^k), H'(u^k) \rangle \leq 0.$$

The problem is to estimate the discrete analog of $d^2 H_c/dt^2$. It turns out that the inequality in (2) is too weak, we need an equation; see Section 2.1 for details. We overcome this difficulty by developing two ideas.

The *first idea* is to identify the elements which are necessary to build an abstract discrete Bakry-Emery method. Unlike in the continuous case, we distinguish between the discrete entropy production $P := -\tau^{-1}(H(u^k) - H(u^{k-1}))$ and the Fisher information $F := \langle A_h(u^k), H'(u^k) \rangle$. We explain this difference in Section 2.

The Bakry-Emery method relies on an estimate of $\tau^{-1}(F(u^k) - F(u^{k-1}))$, which approximates $d^2 H_c/dt^2$. For this estimate, discrete versions of suitable integrations by parts and chain rules are necessary. Our *second idea* is to “translate” a nonlinear integration-by-parts formula to the discrete case, using the systematic integration by parts method of [16]. This leads to a new inequality for numerical three-point schemes as explained next.

Again, consider first the continuous case. We show in Lemma 7 that for all $(A, B) \in R_c := \{(A, B) \in \mathbb{R}^2 : (2A - B - 1)(A + B - 2) < 0\}$ and all smooth positive functions w ,

$$\int_{\mathbb{T}} w_{xx}(w^A)_{xx} w^B dx \geq \kappa_c \int_{\mathbb{T}} w^{A+B-1} w_{xx}^2 dx,$$

where the constant $\kappa_c > 0$ depends on A and B ; see (16) below. The proof is based on systematic integration by parts [16]. The discrete counterpart is the following inequality: For any $0 < \varepsilon \leq 1$, there exists a region R of admissible values (A, B) , containing the line $A = 1$, such that for all $w_0, \dots, w_{N+1} \geq 0$ with $w_N = w_0$, $w_{N+1} = w_1$,

$$(3) \quad \begin{aligned} & \sum_{i=1}^N (w_{i+1} - 2w_i + w_{i-1})(w_{i+1}^A - 2w_i^A + w_{i-1}^A) w_i^B \\ & \geq \kappa \sum_{i=1}^N \min_{j=i, i\pm 1} w_j^{A+B-1} (w_{i+1} - 2w_i + w_{i-1})^2, \end{aligned}$$

where $\kappa = \varepsilon A$; see Lemma 8. Interestingly, the inequality is not true for each term but only with the sum. The admissible set R for (3) is generally smaller than R_c ; see Section 3. We conjecture that $R = R_c$ for $\kappa = 0$.

Inequality (3) is the first nonlinear summation-by-parts formula derived from a systematic method. We believe that this idea will lead to a whole family of new finite-difference inequalities useful in numerical analysis, and we will explore this in a future work.

We apply the abstract discrete Bakry-Emery method in Section 4 to an implicit Euler finite-difference approximation of the porous-medium equation

$$\partial_t u = (u^\beta)_{xx} \quad \text{in } \mathbb{T}, \quad t > 0, \quad u(0) = u^0 \geq 0,$$

where $\beta > 1$ and \mathbb{T} is the one-dimensional torus. We assume, for simplicity, that $\text{meas}(\mathbb{T}) = 1$ and identify \mathbb{T} with $[0, 1]$. The entropy functional is $H_c(u) = \int_{\mathbb{T}} (u^\alpha - \bar{u}^\alpha) dx / (\alpha - 1)$, where $\alpha > 0$ and $\bar{u} = \int_{\mathbb{T}} u^0 dx$ is the constant steady state. We show in Proposition 6 that $H_c(u(t))$ decays exponentially fast to zero for all $(\alpha, \beta) \in S_c$, where

$$(4) \quad S_c = \{(\alpha, \beta) \in \mathbb{R}_+^2 : \alpha + \beta > 1, -2 < \alpha - \beta < 1\},$$

with a decay rate depending on (α, β) and $\min_{\mathbb{T}} u^{\beta-1}$.

To overcome the difficulty with the entropy production inequality, we introduce the new variable $v = u^\alpha$ and write the porous-medium equation in the form

$$(5) \quad \partial_t v = \alpha u^{\alpha-1} \partial_t u = \alpha v^{(\alpha-1)/\alpha} (v^{\beta/\alpha})_{xx}.$$

The advantage of this formulation is that the entropy becomes *linear* in the variable v , thus avoiding inequality (2).

We discretize (5) by an implicit Euler finite-difference scheme. Let $\tau > 0$ be the time step, $h > 0$ the space step, and let $v_i^k = (u_i^k)^\alpha$ be an approximation of $(h^{-1} \int_{(i-1)h}^{ih} u(x, k\tau) dx)^\alpha$, $i = 1, \dots, N$. The iterative scheme reads as

$$(6) \quad v_i^k - v_i^{k-1} = \tau h^{-2} \alpha (v_i^k)^{(\alpha-1)/\alpha} ((v_{i+1}^k)^{\beta/\alpha} - 2(v_i^k)^{\beta/\alpha} + (v_{i-1}^k)^{\beta/\alpha}),$$

where $i = 1, \dots, N$, $k \in \mathbb{N}$, and $v_N^k = v_0^k$, $v_{N+1}^k = v_1^k$. We show in Lemma 10 the existence of solutions to (6) as well as the preservation of nonnegativity. However, the total mass $h \sum_{i=1}^N u_i^k = h \sum_{i=1}^N (v_i^k)^{1/\alpha}$ is not conserved, which is the price that we have to pay for the estimation of the entropy production. We discuss this point in Section 5. Our main result reads as follows.

Theorem 1. *Let $v^k = (v_i^k)$ be a nonnegative solution to (6) and set $u_i^k = (v_i^k)^{1/\alpha}$. Let $0 < \varepsilon < 1$. Then there exist a region $S \subset (0, \infty)^2$, containing the line $\alpha - \beta = 1$, and a number $U > 0$ such that all $(\alpha, \beta) \in S$ with $\alpha > 1$ and $\beta \geq 1$, it holds that*

$$\mathcal{H}(u^k) \leq \mathcal{H}(u^0) e^{-\eta \lambda k \tau}, \quad k \in \mathbb{N},$$

where

$$\mathcal{H}(u^k) = \frac{h}{\alpha - 1} \sum_{i=1}^N ((u_i^k)^\alpha - U^\alpha) dx$$

is the discrete (relative) entropy,

$$\eta = \frac{\log(1 + k\tau)}{k\tau}, \quad \lambda = \frac{8\varepsilon(\alpha - 1)\beta^2}{C_p(\alpha + \beta - 1)^2} \min_{i=1, \dots, N} u_i^{\beta-1},$$

and $C_p = h^2/(4 \sin^2(h\pi)) \geq 1/(4\pi^2)$ the discrete Poincaré constant. Moreover, the total mass $h \sum_{i=1}^N u_i^k$ is increasing in k and converges to U as $k \rightarrow \infty$.

Remark 2 (Exponential versus algebraic decay). The exponential decay rate depends on the minimum of the solution, which is not surprising. Indeed, because of the degeneracy, we cannot generally expect exponential decay; an example is the Barenblatt solution. Algebraic decay rates for implicit Euler finite-volume schemes have been derived in, e.g., [8]. When the minimum is positive, the equation is no longer degenerate, and exponential decay follows. \square

Remark 3 (Shannon entropy). Unfortunately, the theorem does not apply to the Shannon entropy $h \sum_i u_i \log u_i$, corresponding to $\alpha \rightarrow 1$, since $\lambda \rightarrow 0$ as $\alpha \rightarrow 1$. The reason is that for $\alpha \rightarrow 1$, the entropy production P cannot be bounded from above by the Fisher information F and so, Assumption A1 of our abstract Bakry-Emery method does not hold; see Section 2.2. \square

Remark 4 (Discrete gradient flow). Erbar and Maas [12] showed that the gradient flow of the Shannon entropy with respect to a nonlocal transportation measure equals the discrete porous-medium equation in one space dimension. The porous-medium equation in several space dimensions was solved by Benamou et al. [4] by providing a spatial discretization of this equation as a convex optimization problem. In both references, no decay rates have been derived. \square

The set S is illustrated in Figure 1 for two different values of ε . Numerical computations indicate that S converges (in the set theoretical sense) to the set S_c defined in (4) if $\varepsilon \rightarrow 0$ but for fixed $\varepsilon > 0$, S is strictly contained in S_c .

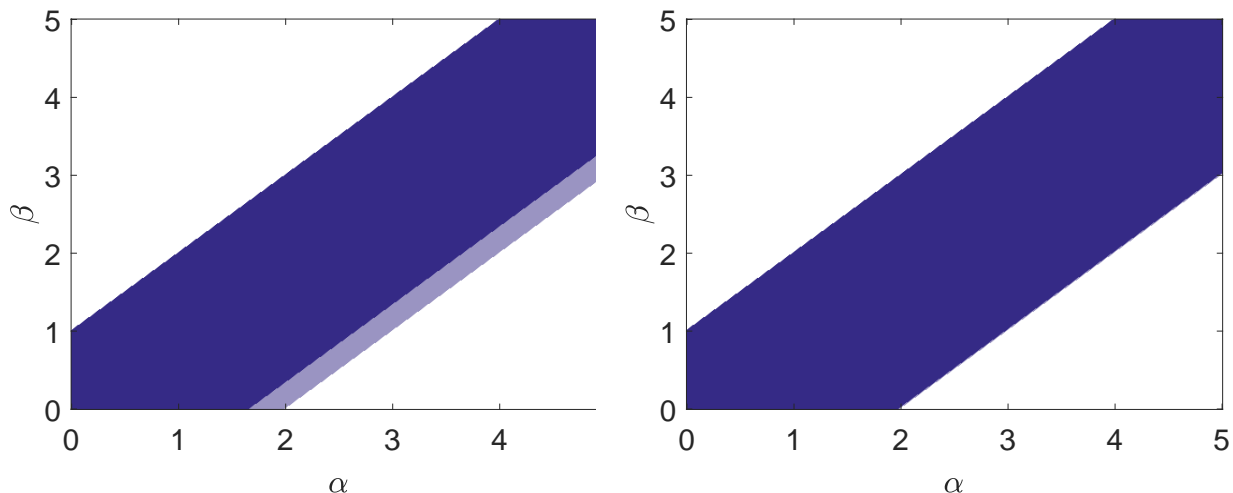


FIGURE 1. Admissible region S for $\varepsilon = 1/4$ (left) and $\varepsilon = 1/100$ (right). The set S_c , defined by $-1 < \alpha - \beta < 2$, is shown in light blue for comparison; it contains the dark blue region S .

The paper is organized as follows. The abstract Bakry-Emery result is presented in Section 2, and in Section 3, inequality (3) is verified. Theorem 1 is proved in Section 4. Numerical examples are presented in Section 5, and some auxiliary inequalities are recalled in the Appendix.

2. AN ABSTRACT BAKRY-EMERY METHOD

In this section, we present our abstract result. In order to identify the key ingredients of the Bakry-Emery method, we recall the basic ideas for continuous evolution equations.

2.1. The continuous Bakry-Emery method. Let us first consider the abstract Cauchy problem

$$(7) \quad \partial_t u + A(u) = 0, \quad t > 0, \quad u(0) = u^0.$$

The nonlinear operator A is defined on some domain $D(A)$ of a Banach space X . We do not specify the properties of A nor its domain since they are not needed in the following. As mentioned in the introduction, the idea of the Bakry-Emery method is to differentiate the entropy functional $H_c : D(A) \rightarrow [0, \infty)$ twice with respect to time along solutions to (7). We define the entropy production $P_c(u(t)) := -\frac{d}{dt} H_c(u(t))$. If $\langle A(u), H'_c(u) \rangle \geq 0$ holds for all $u \in D(A)$ ($\langle \cdot, \cdot \rangle$ is the dual product in X) then

$$P_c(u) = -\langle \partial_t u, H'_c(u) \rangle = \langle A(u), H'_c(u) \rangle \geq 0,$$

i.e., the entropy production is nonnegative and the entropy is nonincreasing along solutions to (7). We call $F_c(u) := \langle A(u), H'_c(u) \rangle$ the generalized Fisher information since if $A(u) = -\Delta u$ on \mathbb{T}^d and $H_c(u) = \int_{\mathbb{T}^d} u(\log u - 1) dx$, we obtain the Fisher information functional $F_c(u) = 4 \int_{\mathbb{T}^d} |\nabla \sqrt{u}|^2 dx$. Clearly, $P_c(u(t)) = F_c(u(t))$ along solutions $u(t)$ to (7).

Differentiating F_c gives

$$\begin{aligned}\frac{dF_c}{dt} &= \langle A'(u)[\partial_t u], H'_c(u) \rangle + \langle A(u), H''_c(u) \partial_t u \rangle \\ &= -\langle A'(u)[A(u)], H'_c(u) \rangle - \langle A(u), H''_c(u) A(u) \rangle,\end{aligned}$$

where $A'(u)$ is the Fréchet derivative of A at u . If the functional inequality

$$(8) \quad \langle A'(u)[A(u)], H'_c(u) \rangle + \langle A(u), H''_c(u) A(u) \rangle \geq \lambda_c \langle A(u), H'_c(u) \rangle$$

holds for some $\lambda_c > 0$ then

$$(9) \quad \frac{dF_c}{dt} \leq -\lambda_c \langle A(u), H'_c(u) \rangle = -\lambda_c F_c,$$

and we conclude exponential decay of $t \mapsto F_c(u(t))$ with rate $\lambda_c > 0$. In particular, $\lim_{t \rightarrow \infty} F_c(u(t)) = 0$. Then, integrating the previous inequality over (t, ∞) , it follows that

$$\frac{dH_c}{dt}(u(t)) = -F_c(u(t)) \leq -\lambda_c \int_t^\infty F_c(u(s)) ds = \lambda_c \int_t^\infty \frac{dH_c}{dt}(u(s)) ds.$$

Assuming that also

$$(10) \quad \lim_{t \rightarrow \infty} H_c(u(t)) = 0,$$

we conclude that

$$\frac{dH_c}{dt}(u(t)) \leq -\lambda_c H_c(u(t)), \quad t > 0,$$

and by Gronwall's lemma, $t \mapsto H_c(u(t))$ converges exponentially fast to zero with rate λ_c .

We see that two assumptions are essential: the functional inequality (8) and the limit (10). On the discrete level, we need to distinguish between the (discrete) entropy production and the (discrete) Fisher information since dH_c/dt and $\langle A(u), H'_c(u) \rangle$ may differ on the discrete level. We assume that both functionals can be estimated by each other. Instead of the functional inequality (8) we assume a discrete version of inequality (9). Finally, a discrete version of (10) is required.

2.2. A discrete Bakry-Emery method. Let two functions $H : \mathbb{R}^N \rightarrow [0, \infty)$ and $F : \mathbb{R}^N \rightarrow [0, \infty)$ be given and define $P(v) := P(v; w) = -\tau^{-1}(H(v) - H(w))$, where $v, w \in \mathbb{R}^N$ and $\tau > 0$. We call H an entropy, F the Fisher information, and P the entropy production. The following result does not need any reference to the solution of a discrete problem.

Proposition 5. *Let $(v^k) \subset \mathbb{R}^N$ be any sequence. We assume that*

A1: *There exist $C_m, C_M > 0$ such that $C_m F(v^k) \leq P(v^k) \leq C_M F(v^k)$ for all $k \in \mathbb{N}$.*

A2: *There exists $\kappa > 0$ such that $F(v^k) - F(v^{k-1}) \leq -\tau \kappa F(v^k)$ for all $k \in \mathbb{N}$.*

A3: $\lim_{k \rightarrow \infty} H(v^k) = 0$.

Then

$$H(v^k) \leq e^{-\eta \lambda k \tau} H(v^0), \quad k \in \mathbb{N},$$

where $\lambda = (C_m/C_M)\kappa$ and $\eta = \log(1 + \tau\lambda)/(\tau\lambda) \in (0, 1)$.

The discrete decay rate λ is generally smaller than the decay rate κ of the Fisher information, since $\eta < 1$ and we may have $C_m < C_M$. If the entropy production and the Fisher information coincide, i.e. $C_m = C_M = 1$, then $\lambda = \kappa$.

Proof. By Assumption A2, it follows that $\lim_{k \rightarrow \infty} F(v^k) = 0$. By Assumption A2 again and the second inequality in Assumption A1, we have

$$F(v^k) - F(v^{k-1}) \leq -\tau \kappa F(v^k) \leq -\tau \kappa C_M^{-1} P(v^k) = \kappa C_M^{-1} (H(v^k) - H(v^{k-1})).$$

Taking the sum from $k = \ell + 1$ to $k = m > \ell + 1$, we find that

$$F(v^m) - F(v^\ell) \leq \kappa C_M^{-1} (H(v^m) - H(v^\ell)).$$

Passing to the limit $m \rightarrow \infty$, observing that $\lim_{m \rightarrow \infty} F(v^m) = 0$ and, by Assumption A3, $\lim_{m \rightarrow \infty} H(v^m) = 0$, we deduce that

$$F(v^\ell) \geq \kappa C_M^{-1} H(v^\ell),$$

which holds for all $\ell \in \mathbb{N}$. It remains to use the first inequality in Assumption A1 to conclude that

$$H(v^k) - H(v^{k-1}) = -\tau P(v^k) \leq -\tau C_m F(v^k) \leq -\tau C_m \kappa C_M^{-1} H(v^k) = -\tau \lambda H(v^k).$$

We deduce that $H(v^k) \leq (1 + \lambda \tau)^{-k} H(v^0) = \exp(-\eta \lambda k \tau) H(v^0)$, finishing the proof. \square

3. A NONLINEAR SUMMATION-BY-PARTS FORMULA

To apply the abstract Bakry-Emery method to the porous-medium equation, we need to verify the assumptions of Proposition 5. The key condition is Assumption A2. To verify it, we “translate” some integrations by parts to the discrete level. It is convenient to investigate the continuous situation first in order to formulate the discrete formula that is needed to show Assumption A2.

Consider the nonlinear diffusion equation

$$(11) \quad \partial_t u = (u^\beta)_{xx}, \quad t > 0, \quad u(0) = u^0 \geq 0 \quad \text{in } \mathbb{T},$$

where $\beta > 0$, and introduce the (relative) entropy

$$H_c(u) = \frac{1}{\alpha - 1} \int_{\mathbb{T}} (u^\alpha - \bar{u}^\alpha) dx, \quad \alpha > 0.$$

Here, $\bar{u} = \int_{\mathbb{T}} u^0(x) dx$ is the constant steady state. (Recall that $\text{meas}(\mathbb{T}) = 1$.)

Proposition 6. *Let $\beta \neq 1$, $\alpha + \beta - 1 > 0$, and $-1 < \alpha - \beta < 2$. Then, for any positive smooth solution to (11),*

$$H_c(u(t)) \leq H_c(u^0) e^{-\lambda_c t}, \quad t > 0,$$

where

$$\lambda_c = \frac{16\pi^2 \alpha \beta \kappa_c}{\alpha + \beta - 1} \min_{\mathbb{T}} u^{\beta-1} \geq 0, \quad \kappa_c = -\frac{4\beta(\alpha - \beta - 2)}{(\alpha + \beta - 1)(\alpha - \beta + 1)} > 0.$$

Proof. Integrating by parts, the time derivatives of $H_c(u(t))$ become

$$\begin{aligned}\frac{dH_c}{dt} &= \frac{\alpha}{\alpha-1} \int_{\mathbb{T}} u^{\alpha-1} (u^\beta)_{xx} dx = -\frac{4\alpha\beta}{(\alpha+\beta-1)^2} \int_{\mathbb{T}} (u^{(\alpha+\beta-1)/2})_x^2 dx, \\ \frac{d^2 H_c}{dt^2} &= -\frac{8\alpha\beta}{(\alpha+\beta-1)^2} \int_{\mathbb{T}} (u^{(\alpha+\beta-1)/2})_x \left(\frac{\alpha+\beta-1}{2} u^{(\alpha+\beta-3)/2} \partial_t u \right)_x dx \\ &= \frac{4\alpha\beta}{\alpha+\beta-1} \int_{\mathbb{T}} u^{(\alpha+\beta-3)/2} (u^{(\alpha+\beta-1)/2})_{xx} (u^\beta)_{xx} dx.\end{aligned}$$

We wish to estimate the second time derivative. To this end, we set $w = u^{(\alpha+\beta-1)/2}$, $A = 2\beta/(\alpha+\beta-1)$, and $B = (\alpha+\beta-3)/(\alpha+\beta-1)$. Then the derivatives can be written as

$$(12) \quad \frac{dH_c}{dt} = -\frac{4\alpha\beta}{(\alpha+\beta-1)^2} \int_{\mathbb{T}} w_x^2 dx, \quad \frac{d^2 H_c}{dt^2} = \frac{4\alpha\beta}{\alpha+\beta-1} \int_{\mathbb{T}} (w^A)_{xx} w_{xx} w^B dx.$$

In Lemma 7 below we show that there exists $\kappa_c > 0$ such that

$$\int_{\mathbb{T}} (w^A)_{xx} w_{xx} w^B dx \geq \kappa_c \int_{\mathbb{T}} w^{A+B-1} w_{xx}^2 dx$$

if the assumption $(2A-B-1)(A+B-2) < 0$ holds. (Note that $\beta \neq 1$ is equivalent to $A+B-2 \neq 0$.) This condition is actually satisfied since

$$(2A-B-1)(A+B-2) = \frac{2(\alpha-\beta-2)(\alpha-\beta+1)}{(\alpha+\beta-1)^2} < 0,$$

and we infer that

$$\frac{d^2 H_c}{dt^2} \geq \frac{4\alpha\beta\kappa_c}{\alpha+\beta-1} \int_{\mathbb{T}} w^{A+B-1} w_{xx}^2 dx = \frac{4\alpha\beta\kappa_c}{\alpha+\beta-1} \int_{\mathbb{T}} u^{\beta-1} (u^{(\alpha+\beta-1)/2})_{xx}^2 dx.$$

Furthermore, by the Poincaré inequality applied to w_x (see Lemma 11),

$$\begin{aligned}\int_{\mathbb{T}} u^{\beta-1} (u^{(\alpha+\beta-1)/2})_{xx}^2 dx &\geq \min_{\mathbb{T}} u^{\beta-1} \int_{\mathbb{T}} (u^{(\alpha+\beta-1)/2})_{xx}^2 dx \\ &\geq 4\pi^2 \min_{\mathbb{T}} u^{\beta-1} \int_{\mathbb{T}} (u^{(\alpha+\beta-1)/2})_x^2 dx = 4\pi^2 \min_{\mathbb{T}} u^{\beta-1} \int_{\mathbb{T}} w_x^2 dx,\end{aligned}$$

and it follows that

$$(13) \quad \frac{d^2 H_c}{dt^2} \geq \frac{16\pi^2\alpha\beta\kappa_c}{\alpha+\beta-1} \min_{\mathbb{T}} u^{\beta-1} \int_{\mathbb{T}} w_x^2 dx = -\lambda_c \frac{dH_c}{dt}.$$

Denoting by $P_c = -dH_c/dt$ the entropy production, this inequality can be formulated as $dP_c/dt \leq -\lambda_c P_c$. Gronwall's lemma then implies that $P_c(u(t)) \leq P_c(u_0)e^{-\lambda_c t}$ for $t > 0$ and in particular $\lim_{t \rightarrow \infty} P_c(u(t)) = 0$.

Integrating (13) over (t, s) with $t < s$ and passing to the limit $s \rightarrow \infty$, we see that

$$\int_0^\infty \int_{\mathbb{T}} w_x^2 dx \leq H_c(u_0) < \infty.$$

Thus, there exists a sequence $t_j \rightarrow \infty$ such that $\|w_x(t_j)\|_{L^2(\mathbb{T})} \rightarrow 0$. Following the arguments of [6, Prop. 1ii],¹ it follows that $\lim_{t_j \rightarrow \infty} H_c(u(t_j)) = 0$, and since $t \mapsto H_c(u(t))$ is nonincreasing, any sequence converges, $\lim_{t \rightarrow \infty} H_c(u(t)) = 0$.

We integrate (13) over (t, ∞) and use $\lim_{t \rightarrow \infty} (dH_c/dt)(u(t)) = \lim_{t \rightarrow \infty} H_c(u(t)) = 0$:

$$-\frac{dH_c}{dt}(u(t)) \geq \lambda_c H_c(u(t)), \quad t > 0.$$

Thus, another application of Gronwall's lemma gives the conclusion. \square

It remains to prove Lemma 7. Set

$$(14) \quad R_c = \{(A, B) \in \mathbb{R}^2 : A > 0, (2A - B - 1)(A + B - 2) < 0\}.$$

Lemma 7. *Let $(A, B) \in R_c$. Then for all smooth positive functions w ,*

$$(15) \quad \int_{\mathbb{T}} w_{xx}(w^A)_{xx} w^B dx \geq \kappa_c \int_{\mathbb{T}} w_{xx}^2 w^{A+B-1} dx,$$

where

$$(16) \quad \kappa_c = \begin{cases} -A(2A - B - 1)/(A + B - 2) > 0 & \text{if } A + B - 2 \neq 0, \\ A & \text{if } A + B - 2 = 0. \end{cases}$$

Proof. The idea of the proof is to employ systematic integration by parts [16]. Since

$$(17) \quad \int_{\mathbb{T}} (w_x^3 w^{A+B-2})_x dx = 0,$$

we can formulate (15) as the following problem: Find $c \in \mathbb{R}$ and $\kappa_c > 0$ such that for all smooth positive functions w ,

$$\int_{\mathbb{T}} (w_{xx}(w^A)_{xx} w^B + c(w_x^3 w^{A+B-3})_x - \kappa_c w_{xx}^2 w^{A+B-1}) dx \geq 0.$$

Calculating the derivatives and setting $\xi_1 = w_x/w$, $\xi_2 = w_{xx}/w$, this inequality is equivalent to

$$\int_{\mathbb{T}} w^{A+B-1} \left((A - \kappa_c) \xi_2^2 + (A^2 - A + 3c) \xi_2 \xi_1^2 + c(A + B - 2) \xi_1^4 \right) dx \geq 0.$$

The idea is to interpret the integrand as a polynomial in the variables ξ_1, ξ_2 and to solve the following polynomial decision problem: Find $c \in \mathbb{R}$ and $\kappa_c > 0$ such that for all $(\xi_1, \xi_2) \in \mathbb{R}^2$,

$$(18) \quad (A - \kappa_c) \xi_2^2 + (A^2 - A + 3c) \xi_2 \xi_1^2 + c(A + B - 2) \xi_1^4 \geq 0.$$

This problem can be solved explicitly. Clearly, it must hold that $A \geq \kappa_c > 0$. We distinguish two cases: $\kappa_c = A$ and $\kappa_c < A$.

First let $0 < \kappa_c < A$. Then (18) is valid if the discriminant is nonpositive,

$$0 \geq (A^2 - A + 3c)^2 - 4c(A - \kappa_c)(A + B - 2)$$

¹Also see the erratum <http://www.asc.tuwien.ac.at/~juengel/publications/pdf/errata05carri.pdf>.

$$\begin{aligned}
&= \left(3c + A(A-1) - \frac{2}{3}(A-\kappa_c)(A+B-2) \right)^2 \\
&\quad - \frac{4}{9}(A-\kappa_c)^2(A+B-2)^2 + \frac{4}{3}A(A-1)(A-\kappa_c)(A+B-2).
\end{aligned}$$

Choosing the minimizing value

$$\begin{aligned}
c &= -\frac{1}{3} \left(A(A-1) - \frac{2}{3}(A-\kappa_c)(A+B-2) \right) \\
(19) \quad &= -\frac{A}{9}(A-2B+1) - \frac{2}{9}\kappa_c(A+B-2),
\end{aligned}$$

the discriminant is nonpositive if and only if

$$\begin{aligned}
0 &\geq -\frac{4}{9}(A-\kappa_c)^2(A+B-2)^2 + \frac{4}{3}A(A-1)(A-\kappa_c)(A+B-2) \\
&= \frac{4}{9}(A-\kappa_c)(A+B-2)(\kappa_c(A+B-2) + A(2A-B-1)).
\end{aligned}$$

Set $\kappa_c = \varepsilon A$ for $0 < \varepsilon < 1$. Then the previous inequality is true if and only if

$$(20) \quad A(A+B-2)(\varepsilon(A+B-2) + 2A-B-1) \leq 0.$$

We infer that if

$$\varepsilon = -\frac{2A-B-1}{A+B-2} > 0$$

then (18) holds. This implies that $\kappa_c = \varepsilon A = -A(2A-B-1)/(A+B-2) > 0$ and we need to choose $A > 0$ and $(2A-B-1)(A+B-2) < 0$.

Next, let $\kappa_c = A$. Then the quadratic term in ξ_2 in (18) vanishes and the mixed term must vanish too, i.e. $c = -A(A-1)/3$. Hence, the coefficient of the remaining term has to be nonnegative, i.e. $-A(A-1)(A+B-2) \geq 0$. If $A = 1$, inequality (15) becomes trivial. The set of all (A, B) such that $A > 0$ and $(A-1)(A+B-2) < 0$ is contained in the set of all (A, B) satisfying $A > 0$ and $(2A-B-1)(A+B-2) < 0$. This finishes the proof. \square

We state now a discrete version of inequality (15).

Lemma 8. *Let $w_0, \dots, w_{N+1} \in \mathbb{R}$ satisfy $w_N = w_0$, $w_{N+1} = w_1$ and let $0 < \varepsilon \leq 1$. There exists a region $R \subset \mathbb{R}^2$, containing the line $A = 1$, such that for all $(A, B) \in R$,*

$$\begin{aligned}
&\sum_{i=1}^N (w_{i+1} - 2w_i + w_{i-1})(w_{i+1}^A - 2w_i^A + w_{i-1}^A)w_i^B \\
(21) \quad &\geq \kappa \sum_{i=1}^N \min_{j=i, i\pm 1} w_j^{A+B-1} (w_{i+1} - 2w_i + w_{i-1})^2,
\end{aligned}$$

where $\kappa = \varepsilon A > 0$.

The lemma is trivial as stated since (21) clearly holds for $R = \{(A, B) : A = 1\}$ with $\kappa = 1$. Figure 2 illustrates the numerical admissible regions for (A, B) for two different

values of ε . The admissible region R is smaller than the region R_c for the continuous case but it approaches the latter region as $\kappa \rightarrow 0$.

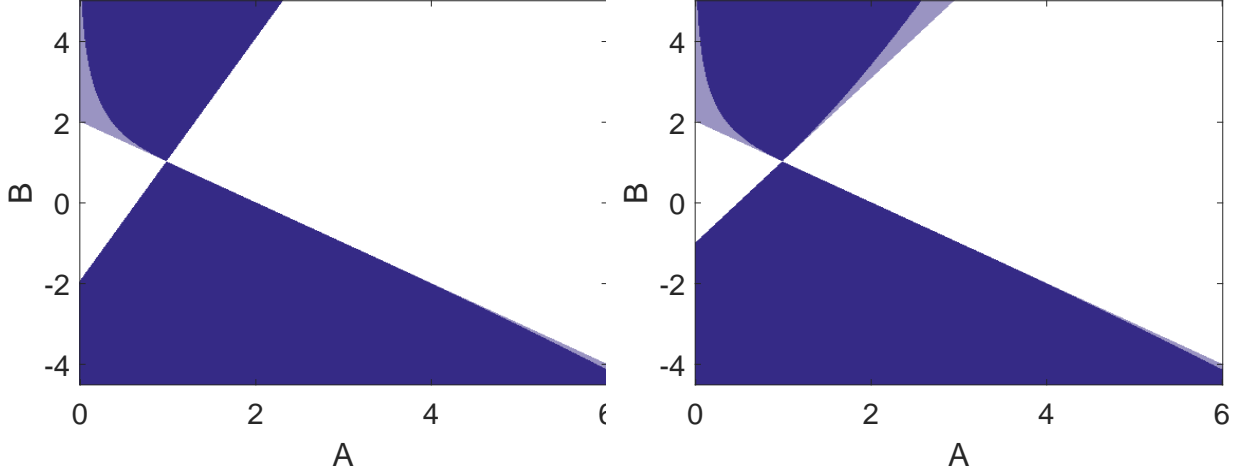


FIGURE 2. The regions of admissible (A, B) such that $T(X, Y) \geq 0$ for all $X, Y \geq 0$ using c as in (19) with $\kappa_c = \kappa$ and $\kappa = A/4$ (left), $\kappa = A/100$ (right). The set R is depicted in dark blue, $R_c \supset R$ in light blue.

The idea of the proof of (21) is to add the following discrete version of the integration-by-parts formula (17), namely

$$\frac{1}{\rho^3} \sum_{i=1}^N \left(M(w_{i+1}, w_i)^{A+B+1-3\rho} (w_{i+1}^\rho - w_i^\rho)^3 - M(w_i, w_{i-1})^{A+B+1-3\rho} (w_i^\rho - w_{i-1}^\rho)^3 \right) = 0.$$

The sum vanishes because of the periodic boundary conditions. Here $\rho > 0$ is a free parameter, and the function $M(x, y)$ is a symmetric mean value, i.e., it satisfies

$$(22) \quad M(x, y) = M(y, x), \quad M(\lambda x, \lambda y) = \lambda M(x, y), \quad M(x, x) = x$$

for all $x, y, \lambda \geq 0$. For the numerical simulations below, we will choose $\rho = (A + B + 1)/3$ such that the mean function does not need to be specified. Then (21) holds if we can show the following inequality for all admissible (A, B) and $w_i \neq 0$:

$$\begin{aligned} & \sum_{i=1}^N w_i^{A+B+1} \left\{ \left(\left(\frac{w_{i+1}}{w_i} \right)^A + \left(\frac{w_{i-1}}{w_i} \right)^A - 2 \right) \left(\frac{w_{i+1}}{w_i} + \frac{w_{i-1}}{w_i} - 2 \right) \right. \\ & \quad - \kappa \min_{j=i, i \pm 1} \left(\frac{w_j}{w_i} \right)^{A+B-1} \left(\frac{w_{i+1}}{w_i} + \frac{w_{i-1}}{w_i} - 2 \right)^2 \\ & \quad + \frac{c}{\rho^3} \left(M \left(\frac{w_{i+1}}{w_i}, 1 \right)^{A+B+1-3\rho} \left(\left(\frac{w_{i+1}}{w_i} \right)^\rho - 1 \right)^3 \right. \\ & \quad \left. \left. - M \left(\frac{w_{i-1}}{w_i}, 1 \right)^{A+B+1-3\rho} \left(1 - \left(\frac{w_{i-1}}{w_i} \right)^\rho \right)^3 \right) \right\} \geq 0. \end{aligned}$$

We verify this inequality pointwise, i.e. setting $X = w_{i+1}/w_i$ and $Y = w_{i-1}/w_i$, we wish to find $c \in \mathbb{R}$, $\kappa > 0$ such that for all $X, Y > 0$,

$$(23) \quad \begin{aligned} T(X, Y) &:= (X^A + Y^A - 2)(X + Y - 2) \\ &+ \frac{c}{\rho^3} \left(M(X, 1)^{A+B+1-3\rho} (X^\rho - 1)^3 + M(Y, 1)^{A+B+1-3\rho} (Y^\rho - 1)^3 \right) \\ &- \kappa \min\{1, X^{A+B-1}, Y^{A+B-1}\} (X + Y - 2)^2 \geq 0. \end{aligned}$$

The first term $(X^A + Y^A - 2)(X + Y - 2)$ becomes negative in certain regions; see Figure 3. It is compensated by the second term (shift term) on the right-hand side of (23) if we choose the constant c according to (19) with $\kappa = \kappa_c$ as in (16).

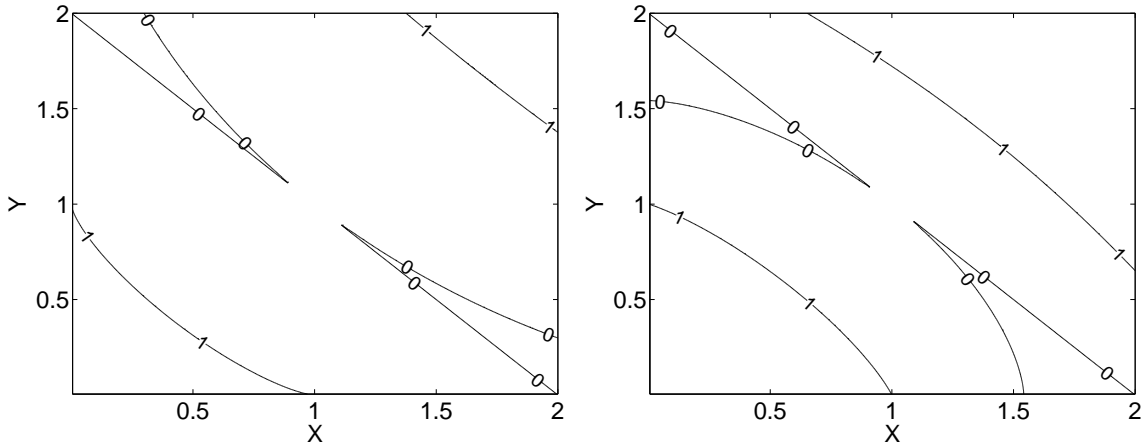


FIGURE 3. Level sets $(X^A + Y^A - 2)(X + Y - 2) = 0$ and $(X^A + Y^A - 2)(X + Y - 2) = 1$ for $A = 0.6$, $B = 4$ (left) and $A = 1.6$, $B = 2.5$ (right). We have chosen $\kappa = \kappa_0 = A/200$ and c as in (19).

Unfortunately, it seems to be difficult to prove (23) analytically in full generality. Note that polynomial quantifier elimination does not apply if A and B are not integers, since the function $T(X, Y)$ generally is *not* a polynomial. Instead, we verify (23) analytically for all $(A, B) \in R_c$ and all (X, Y) in some neighborhood of $(1, 1)$.

Lemma 9. *Let T be given by (23) and let $(A, B) \in R_c$, where R_c is defined in (14). Then there exists a neighborhood W of $(1, 1)$ such that for all $(X, Y) \in W$,*

$$T(X, Y) \geq 0$$

holds for c as in (19) and with $\kappa_c = \kappa$ as in (16).

If the step size $h > 0$ is small enough, we expect that the quotients w_{i+1}/w_i are close to one for all $i = 0, \dots, N - 1$. This means that (X, Y) lies in a neighborhood of $(1, 1)$, and the lemma applies.

Proof. We use the local coordinates $u = (X + Y - 2)/h^2$ and $v = (X - Y)/(2h)$, which correspond to (central) second-order and first-order derivatives. Then $X = 1 + hv + h^2u/2$

and $Y = 1 - hv + h^2u/2$. We develop T as a function of h at $h = 0$. For this, we observe that $M(1, 1) = 1$ and $M_X(1, 1) = M_Y(1, 1) = 1/2$. Indeed, we infer from the properties (22) that

$$\begin{aligned}
M_X(1, 1) &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (M(1 + \varepsilon, 1) - M(1, 1)) = \lim_{\varepsilon \rightarrow 0} \left(\frac{1 + \varepsilon}{\varepsilon} M\left(1, \frac{1}{1 + \varepsilon}\right) - \frac{1}{\varepsilon} \right) \\
&= \lim_{\varepsilon \rightarrow 0} \left(\frac{1 + \varepsilon}{\varepsilon} M\left(1, 1 - \frac{\varepsilon}{1 + \varepsilon}\right) - \frac{1 + \varepsilon}{\varepsilon} + 1 \right) \\
&= \lim_{\varepsilon \rightarrow 0} \left(\frac{1 + \varepsilon}{\varepsilon} M\left(1 - \frac{\varepsilon}{1 + \varepsilon}, 1\right) - \frac{1 + \varepsilon}{\varepsilon} + 1 \right) \\
&= \lim_{\delta \rightarrow 0} \frac{1}{\delta} (M(1 - \delta, 1) - M(1, 1)) + 1 \\
&= -M_X(1, 1) + 1,
\end{aligned}$$

which implies that $M_X(1, 1) = 1/2$.

Calculating the Taylor series of the terms in T with respect to h at $h = 0$ leads to

$$\begin{aligned}
(X^A + Y^A - 2)(X + Y - 2) &= Au((A - 1)v^2 + u)h^4 + O(h^6), \\
\frac{c}{\rho^3} M(X, 1)^{A+B+1-3\rho} (X^\rho - 1)^3 &= cv^3h^3 + \frac{c}{2}((A + B - 2)v^2 + 3u)v^2h^4 + O(h^5), \\
\frac{c}{\rho^3} M(Y, 1)^{A+B+1-3\rho} (Y^\rho - 1)^3 &= -cv^3h^3 + \frac{c}{2}((A + B - 2)v^2 + 3u)v^2h^4 + O(h^5).
\end{aligned}$$

In particular, as expected, the explicit choices of both ρ and $M(x, y)$ do not change the behavior of the shift term locally around the equilibrium $w_{i-1} = w_i = w_{i+1}$ or $h = 0$. Moreover, $\min\{1, X^{A+B-1}, Y^{A+B-1}\} = 1 + O(h)$ and $(X + Y - 2)^2 = u^2h^4$. Combining these expressions gives

$$T(X, Y) = h^4 \left((A - \kappa)u^2 + (A(A - 1) + 3c)uv^2 + c(A + B - 2)v^4 \right) + O(h^5).$$

The polynomial

$$(u, v) \mapsto (A - \kappa)u^2 + (A(A - 1) + 3c)uv^2 + c(A + B - 2)v^4$$

is the same as in (18). The proof of Lemma 7 shows that it is nonnegative for all $(A, B) \in R_c$ with c as in (19) and κ_c as in (16). We deduce that $T(X, Y) \geq 0$ holds for all $(A, B) \in R_c$ if $h \in \mathbb{R}$ is sufficiently small. This proves the lemma. \square

4. DISCRETE POROUS-MEDIUM EQUATION

We apply the abstract Bakry-Emery method to a finite-difference approximation of the porous-medium equation, i.e., we choose $A(u) = -(u^\beta)_{xx}$ on \mathbb{T} for suitable functions u . Let $\tau > 0$ be the time step and $h > 0$ the space step. A natural scheme would be

$$u_i^k - u_i^{k-1} = \tau h^{-2} \left((u_{i+1}^k)^\beta - 2(u_i^k)^\beta + (u_{i-1}^k)^\beta \right),$$

for all $i = 1, \dots, N$, $k \in \mathbb{N}$, and $u_N^k = u_0^k$, $u_{N+1}^k = u_1^k$. The corresponding discrete entropy is $H(u^k) = h \sum_{i=1}^N ((u_i^k)^\alpha - \bar{u}^\alpha)/(\alpha - 1)$ and $\bar{u} = h \sum_{i=1}^N u_i^0$ is the constant steady state. We choose $\alpha > 1$ and $\beta > 1$.

Unfortunately, the abstract Bakry-Emery method cannot be applied to this scheme. The problem is the second inequality in Assumption A1. Indeed, using the inequality $y^\alpha - z^\alpha \geq \alpha z^{\alpha-1}(y - z)$ for all $y, z \geq 0$, which follows from the convexity of $z \mapsto z^\alpha$ for $\alpha > 1$, inserting the numerical scheme and then summing by parts, we find that

$$\begin{aligned} -\tau P &= H(u^k) - H(u^{k-1}) = h \sum_{i=0}^N ((u_i^k)^\alpha - (u_i^{k-1})^\alpha) \\ &\geq \alpha h \sum_{i=1}^N (u_i^{k-1})^{\alpha-1} (u_i^k - u_i^{k-1}) \\ &= \alpha h^{-1} \tau \sum_{i=1}^N (u_i^{k-1})^{\alpha-1} \left(((u_{i+1}^k)^\beta - (u_i^k)^\beta) - ((u_i^k)^\beta - (u_{i-1}^k)^\beta) \right) \\ &= -\alpha h^{-1} \sum_{i=1}^N ((u_{i+1}^{k-1})^{\alpha-1} - (u_i^{k-1})^{\alpha-1}) ((u_{i+1}^k)^\beta - (u_i^k)^\beta). \end{aligned}$$

This expression cannot be estimated further; it may even have the wrong sign. We need a scheme that avoids the use of the inequality $y^\alpha - z^\alpha \geq \alpha z^{\alpha-1}(y - z)$. We stress the fact that this problem does not occur in the semi-discrete scheme

$$\partial_t u_i = h^{-2} ((u_{i+1}^k)^\beta - 2(u_i^k)^\beta + (u_{i-1}^k)^\beta),$$

since then

$$\begin{aligned} \frac{dH}{dt} &= \frac{\alpha h}{\alpha - 1} \sum_{i=0}^N u_i^{\alpha-1} \partial_t u_i = \frac{\alpha}{(\alpha - 1)h} \sum_{i=0}^N u_i^{\alpha-1} ((u_{i+1}^k)^\beta - 2(u_i^k)^\beta + (u_{i-1}^k)^\beta) \\ &= -\frac{\alpha}{(\alpha - 1)h} \sum_{i=0}^N ((u_{i+1}^k)^{\alpha-1} - (u_i^k)^{\alpha-1}) ((u_{i+1}^k)^\beta - (u_i^k)^\beta), \end{aligned}$$

and this expression is nonpositive (since $\alpha > 1$).

Our idea is to make the entropy production *linear* in its argument. For this, we introduce the new variable $v_i^k = (u_i^k)^\alpha$. In the (continuous) variable $v = u^\alpha$, the evolution equation transforms to $\partial_t v = -v^{(\alpha-1)/\alpha} \Delta(v^{\beta/\alpha})$, which inspires the numerical scheme

$$(24) \quad v_i^k - v_i^{k-1} = \alpha \tau h^{-2} (v_i^k)^{(\alpha-1)/\alpha} ((v_{i+1}^k)^{\beta/\alpha} - 2(v_i^k)^{\beta/\alpha} + (v_{i-1}^k)^{\beta/\alpha}),$$

for all $i = 1, \dots, N$, $k \in \mathbb{N}$, and $v_N^k = v_0^k$, $v_{N+1}^k = v_1^k$. The discrete entropy and Fisher information become

$$H(v^k) = \frac{h}{\alpha - 1} \sum_{i=1}^N (v_i^k - V), \quad F(v^k) = \frac{1}{h} \sum_{i=1}^N ((v_{i+1}^k)^\gamma - (v_i^k)^\gamma)^2,$$

where $V > 0$ has to be determined and $\gamma = (\alpha + \beta - 1)/(2\alpha)$. The entropy production can be estimated, using summation by parts, as

$$\begin{aligned}
 -\tau P(v^k) &= H(v^k) - H(v^{k-1}) = \frac{h}{\alpha - 1} \sum_{i=1}^N (v_i^k - v_i^{k-1}) \\
 &= \frac{\alpha\tau}{(\alpha - 1)h} \sum_{i=1}^N (v_i^k)^{(\alpha-1)/\alpha} ((v_{i+1}^k)^{\beta/\alpha} - 2(v_i^k)^{\beta/\alpha} + (v_{i-1}^k)^{\beta/\alpha}) \\
 (25) \quad &= -\frac{\alpha\tau}{(\alpha - 1)h} \sum_{i=1}^N ((v_{i+1}^k)^{(\alpha-1)/\alpha} - (v_i^k)^{(\alpha-1)/\alpha}) ((v_{i+1}^k)^{\beta/\alpha} - (v_i^k)^{\beta/\alpha}) \leq 0.
 \end{aligned}$$

According to Lemma 13, the entropy production can be estimated from below *and* above in terms of the Fisher information.

After this motivation, we prove the existence of solutions to (24).

Lemma 10. *For given $v_i^{k-1} \geq 0$, $i = 1, \dots, N$, there exists a solution $v_i^k \geq 0$, $i = 1, \dots, N$, to (24).*

Proof. We give only a sketch of the proof since the existence of solutions follows from a standard fixed-point theorem. We only provide the a priori estimates needed for this argument. First multiply (24) by $(v_i^k)_- = \min\{v_i^k, 0\}$ and sum over $i = 1, \dots, N$. Since $v_i^k(v_i^k)_- = (v_i^k)_-^2$, we obtain

$$\begin{aligned}
 \sum_{i=1}^N (v_i^k)_-^2 &= \sum_{i=1}^N v_i^{k-1} (v_i^k)_- + \alpha\tau h^{-2} \sum_{i=1}^N (v_i^k)_-^{2-1/\alpha} ((v_{i+1}^k)^{\beta/\alpha} - 2(v_i^k)^{\beta/\alpha} + (v_{i-1}^k)^{\beta/\alpha}) \\
 &\leq \alpha\tau h^{-2} \sum_{i=1}^N (v_i^k)_-^{2-1/\alpha} ((v_{i+1}^k)^{\beta/\alpha} - (v_i^k)^{\beta/\alpha}) - ((v_i^k)^{\beta/\alpha} - (v_{i-1}^k)^{\beta/\alpha}).
 \end{aligned}$$

By summation by parts, this becomes

$$\sum_{i=1}^N (v_i^k)_-^2 \leq -\alpha\tau h^{-2} \sum_{i=1}^N ((v_{i+1}^k)_-^{2-1/\alpha} - (v_i^k)_-^{2-1/\alpha}) ((v_{i+1}^k)^{\beta/\alpha} - (v_i^k)^{\beta/\alpha}) \leq 0,$$

since $z \mapsto z_-^{2-1/\alpha}$ is nondecreasing. We infer that $(v_i^k)_- = 0$ and hence $v_i^k \geq 0$. Next, by (25),

$$\sum_{i=1}^N v_i^k \leq \sum_{i=1}^N v_i^{k-1} \leq \sum_{i=1}^N v_i^0,$$

and this is the desired a priori estimate. \square

Next, we turn to the proof of our main result.

Proof of Theorem 1. We verify the assumptions of Proposition 5. For this, we continue our estimates for P . Applying Lemma 13 with $a = (\alpha - 1)/\alpha$ and $b = \beta/\alpha$ to (25), we obtain

the inequalities

$$P(v^k) \leq \frac{\alpha}{\alpha-1} F(v^k), \quad F(v^k) \leq \frac{\alpha\gamma^2}{\beta} P(v^k) = \frac{(\alpha+\beta-1)^2}{4\alpha\beta} P(v^k).$$

Thus, Assumption A1 is satisfied with $C_m = 4\alpha\beta/(\alpha+\beta-1)^2$ and $C_M = \alpha/(\alpha-1)$.

Next, we estimate the difference $F(v^k) - F(v^{k-1})$. To this end, we set $v_i := v_i^k$, $\bar{v}_i := v_i^{k-1}$, $a_i := (v_i - \bar{v}_i)/\tau$ and write

$$\begin{aligned} F(v) - F(\bar{v}) &= \frac{1}{h} \sum_{i=1}^N ((v_{i+1}^\gamma - v_i^\gamma)^2 - (\bar{v}_{i+1}^\gamma - \bar{v}_i^\gamma)^2) \\ &= \frac{1}{h} \sum_{i=1}^N \left((v_{i+1}^\gamma - v_i^\gamma)^2 - ((v_{i+1} - \tau a_{i+1})^\gamma - (v_i - \tau a_i)^\gamma)^2 \right) \\ &=: G(\tau). \end{aligned}$$

The idea of the proof is to expand $G(\tau)$ around zero:

$$F(v) - F(\bar{v}) = G(0) + G'(0)\tau + \frac{1}{2}G''(\xi)\tau^2$$

for some $\xi \in (0, \tau)$. We show that the right-hand side can be bounded from above by $-\tau K F(v)$ for some $K > 0$, which verifies Assumption A2. This idea has been first employed in [8]. Clearly, we have $G(0) = 0$. The first derivatives of G equal

$$\begin{aligned} G'(\tau) &= 2\gamma h^{-1} \sum_{i=1}^N ((v_{i+1} - \tau a_{i+1})^\gamma - (v_i - \tau a_i)^\gamma) \\ &\quad \times ((v_{i+1} - \tau a_{i+1})^{\gamma-1} a_{i+1} - (v_i - \tau a_i)^{\gamma-1} a_i), \\ G''(\tau) &= -2\gamma h^{-1} \sum_{i=1}^N \left(\gamma ((v_{i+1} - \tau a_{i+1})^{\gamma-1} a_{i+1} - (v_i - \tau a_i)^{\gamma-1} a_i)^2 \right. \\ &\quad \left. + (\gamma-1)((v_{i+1} - \tau a_{i+1})^\gamma - (v_i - \tau a_i)^\gamma) \right. \\ &\quad \left. \times ((v_{i+1} - \tau a_{i+1})^{\gamma-2} a_{i+1}^2 - (v_i - \tau a_i)^{\gamma-2} a_i^2) \right). \end{aligned}$$

First, we claim that $G''(\tau) \leq 0$ for any $\tau > 0$. Indeed, we replace $v_i - \tau a_i$ by \bar{v}_i and obtain

$$\begin{aligned} G''(\tau) &= -2\gamma h^{-1} \sum_{i=0}^N (c_1 a_{i+1}^2 + c_2 a_{i+1} a_i + c_3 a_i^2), \quad \text{where} \\ c_1 &= \gamma \bar{v}_{i+1}^{2\gamma-2} + (\gamma-1) \bar{v}_{i+1}^{\gamma-2} (\bar{v}_{i+1}^\gamma - \bar{v}_i^\gamma) = (2\gamma-1) \bar{v}_{i+1}^{2\gamma-2} - (\gamma-1) \bar{v}_{i+1}^{\gamma-2} \bar{v}_i^\gamma, \\ c_2 &= -2\gamma \bar{v}_{i+1}^{\gamma-1} \bar{v}_i^{\gamma-1}, \\ c_3 &= \gamma \bar{v}_i^{2\gamma-2} - (\gamma-1) \bar{v}_i^{\gamma-2} (\bar{v}_{i+1}^\gamma - \bar{v}_i^\gamma) = (2\gamma-1) \bar{v}_i^{2\gamma-2} - (\gamma-1) \bar{v}_i^{\gamma-2} \bar{v}_{i+1}^\gamma. \end{aligned}$$

It holds that $c_1 \geq 0$, since this inequality is equivalent to $(2\gamma - 1)\bar{v}_{i+1}^\gamma \geq (\gamma - 1)\bar{v}_i^\gamma$, and this is true for $1/2 \leq \gamma \leq 1$ (which is equivalent to $\beta \geq 1$ and $\alpha - \beta \geq -1$). Moreover, the discriminant $4c_1c_3 - c_2^2 \geq 0$ is equivalent to

$$4(2\gamma - 1)(1 - \gamma)(\bar{v}_{i+1}\bar{v}_i)^{\gamma-2}(\bar{v}_{i+1}^\gamma - \bar{v}_i^\gamma)^2 \geq 0,$$

which also holds true for $1/2 \leq \gamma \leq 1$. This shows that $G''(\tau) \leq 0$ and consequently,

$$F(v) - F(\bar{v}) = G(\tau) = G(0) + \tau G'(0) + \frac{\tau^2}{2} G''(\xi) \leq \tau G'(0).$$

It remains to compute $G'(0)$. Inserting the definition of a_i , we find that

$$\begin{aligned} G'(0) &= 2\gamma h^{-1} \sum_{i=1}^N (v_{i+1}^\gamma - v_i^\gamma)(v_{i+1}^{\gamma-1} a_{i+1} - v_i^{\gamma-1} a_i) \\ &= -2\gamma h^{-1} \sum_{i=1}^N v_i^{\gamma-1} a_i (v_{i+1}^\gamma - 2v_i^\gamma + v_{i-1}^\gamma) \\ &= -2\alpha\gamma h^{-3} \sum_{i=1}^N v_i^{(\alpha+\beta-3)/(2\alpha)} (v_{i+1}^{\beta/\alpha} - 2v_i^{\beta/\alpha} + v_{i-1}^{\beta/\alpha}) (v_{i+1}^\gamma - 2v_i^\gamma + v_{i-1}^\gamma), \end{aligned}$$

since $v_i^{\gamma-1} v_i^{(\alpha-1)/\alpha} = v_i^{(\alpha+\beta-3)/(2\alpha)}$.

We apply Lemma 8 with $w_i = v_i^\gamma$, $A = 2\beta/(\alpha + \beta - 1)$ and $B = (\alpha + \beta - 3)/(\alpha + \beta - 1)$ and infer that

$$G'(0) \leq -2\alpha\gamma\kappa h^{-3} \sum_{i=1}^N \min_{j=i, i\pm 1} v_j^{(\beta-1)/\alpha} (v_{i+1}^\gamma - 2v_i^\gamma + v_{i-1}^\gamma)^2.$$

By the discrete Poincaré-Wirtinger inequality (Lemma 12), applied with $z_i = v_{i+1}^\gamma - v_i^\gamma$, it follows that

$$\begin{aligned} G'(0) &\leq -2C_p^{-1}\alpha\gamma\kappa h^{-1} \min_{i=1, \dots, N} v_i^{(\beta-1)/\alpha} \sum_{i=1}^N (v_{i+1}^\gamma - v_i^\gamma)^2 \\ &= -2C_p^{-1}\alpha\gamma\kappa \min_{i=1, \dots, N} v_i^{(\beta-1)/\alpha} F(v), \end{aligned}$$

and hence, with $\kappa_0 = 2C_p^{-1}\alpha\gamma\kappa \min_{i=1, \dots, N} v_i^{(\beta-1)/\alpha}$,

$$F(v) - F(\bar{v}) \leq -\tau\kappa_0 F(v).$$

This shows Assumption A2 of Proposition 5 and, in particular, after applying Gronwall's lemma, $\lim_{k \rightarrow \infty} F(v^k) = 0$.

It remains to prove that Assumption A3, i.e. $\lim_{k \rightarrow \infty} H(v^k) = 0$, holds. We know that

$$v_i^k \leq \sum_{j=1}^N v_j^k \leq \sum_{j=1}^N v_j^0 < \infty,$$

so, for any fixed $i = 1, \dots, N$, (v_i^k) is bounded. Therefore, there exists a sequence $k_j \rightarrow \infty$ such that $v_i^{k_j} \rightarrow y_i$ for some $y_i \geq 0$. By the discrete Poincaré-Wirtinger inequality (Lemma 12), applied to $z_i = (v_i^k)^\gamma - (V^k)^\gamma$, where $(V^k)^\gamma := h \sum_{i=1}^N (v_i^k)^\gamma$, it follows that

$$\sum_{i=1}^N ((v_i^k)^\gamma - (V^k)^\gamma)^2 \leq C_p h^{-2} \sum_{i=1}^N ((v_{i+1}^k)^\gamma - (v_i^k)^\gamma)^2 = C_p h^{-1} F(v^k).$$

Since $\lim_{k \rightarrow \infty} F(v^k) = 0$, we deduce that $(v_i^{k_j})$ and V^{k_j} have the same limit, say $y := y_i$. Set $U := y^{1/\alpha}$. This defines the entropy

$$\mathcal{H}(u^k) = \frac{h}{\alpha - 1} \sum_{i=1}^N ((u_i^k)^\alpha - U^\alpha)$$

for $u_i^k := (v_i^k)^{1/\alpha}$. It holds that $\mathcal{H}(u^{k_j}) \rightarrow 0$ as $k_j \rightarrow \infty$. But $k \mapsto \mathcal{H}(u^k)$ is nonincreasing, from which we deduce that $H(v^k) = \mathcal{H}(u^k) \rightarrow 0$ for any sequence $k \rightarrow \infty$.

According to Proposition 5, the discrete entropy converges exponentially with decay rate

$$\lambda = \frac{C_m}{C_M} \kappa_0 = \frac{4(\alpha - 1)\beta}{\alpha + \beta - 1} \frac{\kappa}{C_p} \min_{i=1, \dots, N} u_i^{\beta-1} = \frac{8\varepsilon(\alpha - 1)\beta^2}{C_p(\alpha + \beta - 1)^2} \min_{i=1, \dots, N} u_i^{\beta-1}.$$

Next, we claim that the total mass $h \sum_{i=1}^N u_i^k$ is nondecreasing in k . Indeed, by the concavity of $z \mapsto z^{1/\alpha}$ (recall that $\alpha > 1$), we have $y^{1/\alpha} - z^{1/\alpha} \geq (1/\alpha)y^{(1-\alpha)/\alpha}(y - z)$ for all $y, z \geq 0$ and hence,

$$\sum_{i=1}^N (u_i^k - u_i^{k-1}) = \sum_{i=1}^N ((v_i^k)^{1/\alpha} - (v_i^{k-1})^{1/\alpha}) \geq \frac{1}{\alpha} \sum_{i=1}^N (v_i^k)^{(1-\alpha)/\alpha} (v_i^k - v_i^{k-1}).$$

Inserting scheme (6), we find that

$$\sum_{i=1}^N (u_i^k - u_i^{k-1}) \geq \frac{\tau}{h^2} \sum_{i=1}^N ((v_{i+1}^k)^{\beta/\alpha} - 2(v_i^k)^{\beta/\alpha} + (v_{i-1}^k)^{\beta/\alpha}) = 0,$$

since v_i^k satisfies periodic boundary conditions. This shows the claim.

The monotonicity of the total mass and the convergence property $h \sum_{i=1}^N u_i^{k_j} \rightarrow y^{1/\alpha} = U$ as $k_j \rightarrow \infty$ imply that $h \sum_{i=1}^N u_i^k \rightarrow U$ for $k \rightarrow \infty$, and the convergence is monotone. This finishes the proof. \square

5. NUMERICAL EXAMPLES

We present some numerical results for the porous-medium equation discretized in the previous section. As initial datum we choose the Barenblatt profile

$$u^0(x) = \frac{1}{t_0^{1/(\beta+1)}} \left(C - \frac{\beta-1}{2\beta} \frac{|x-x_0|^2}{t_0^{2/(\beta+1)}} \right)_+^{1/(\beta-1)},$$

where $z_+ = \max\{0, z\}$. We consider two cases. For the slow diffusion case $\beta = 4$, we choose $x_0 = 0.5$, $t_0 = 10^{-4}$, and

$$C = \frac{\beta - 1}{2\beta} \frac{|x_0|^2}{(t_{\text{end}} + t_0)^{2/(\beta+1)}}, \quad t_{\text{end}} = 5 \cdot 10^{-4}.$$

The profile will hit the boundary of $\Omega = (0, 1)$ at time t_{end} . For the fast diffusion case $\beta = 0.5$, we take $x_0 = 0.5$, $t_0 = 10^{-2}$, and $C = t_0^{(\beta-1)/(\beta+1)}$ such that the maximum of the initial profile equals 1.

Figure 4 illustrates the evolution of the total mass for $\alpha = 2$, $\beta = 0.5$ (left) and $\alpha = 3$, $\beta = 4$ (right). As predicted in Theorem 1, the total mass is indeed increasing in time. The mass defect scales well with both the time step τ and the grid size h , where the influence of τ is more prevalent.

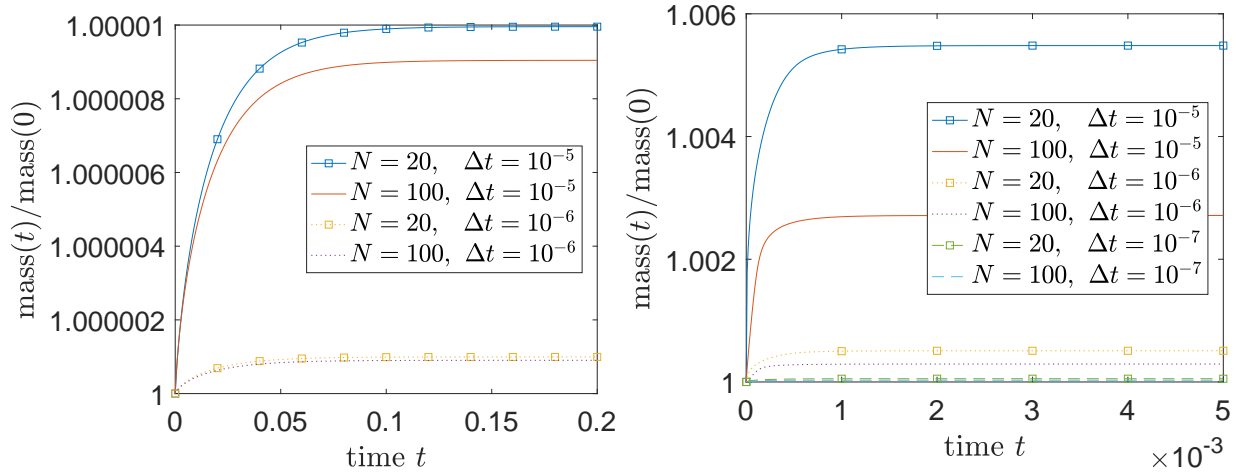


FIGURE 4. Evolution of the total mass for two test scenarios (left: $\alpha = 2$, $\beta = 0.5$, right: $\alpha = 3$, $\beta = 4$).

The time decay of the (relative) entropy \mathcal{H} is shown in Figure 5 for various space and time steps. We observe that the decay is indeed exponential. Here, the steady state u_∞ (which is needed to define the relative entropy) is given by $u_\infty = h \sum_{i=0}^N u_i^{k_{\text{max}}}$, where k_{max} is the final time step. This choice clearly depends on the scheme since the mass is not conserved. The relative entropy converges exponentially even when (α, β) is chosen outside of the admissible region; see Figure 6.

APPENDIX A. AUXILIARY INEQUALITIES

Lemma 11 (Poincaré-Wirtinger inequality). *Let $\text{meas}(\mathbb{T}) = 1$. It holds for all $v \in H^1(\mathbb{T})$ satisfying $\int_{\mathbb{T}} u dx = 0$ that*

$$\int_{\mathbb{T}} u^2 dx \leq C_P \int_{\mathbb{T}} u_x^2 dx,$$

and the constant $C_P = 1/(4\pi^2)$ is sharp.

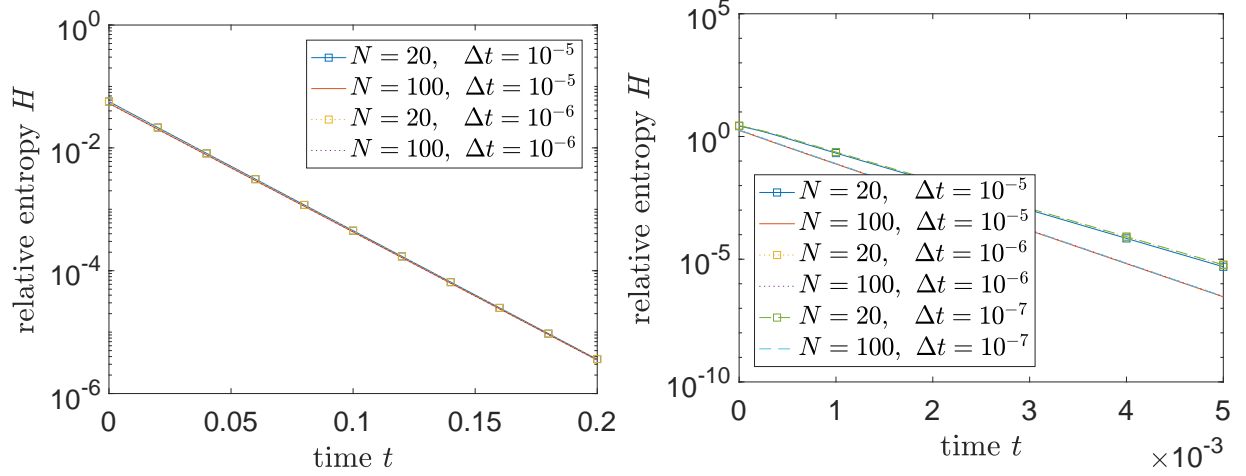


FIGURE 5. Evolution of the relative entropy for two test scenarios in the admissible region (left: $\alpha = 2, \beta = 0.5$, right: $\alpha = 3, \beta = 4$).

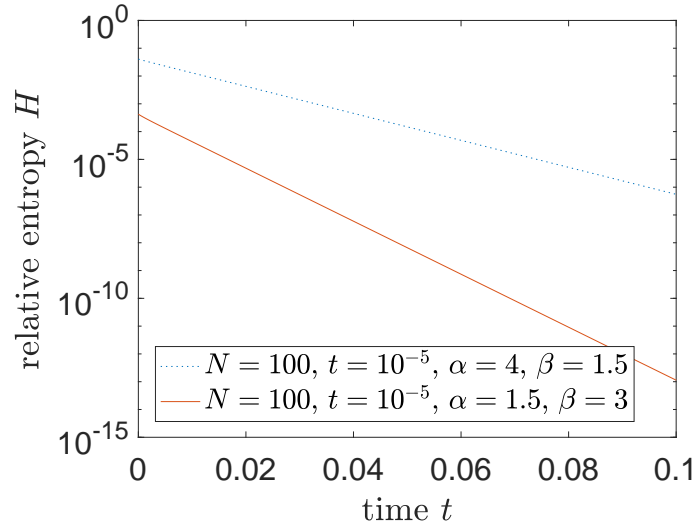


FIGURE 6. Evolution of the relative entropies for (α, β) outside of the admissible region.

Lemma 12 (Discrete Poincaré-Wirtinger inequality). *Let $N \in \mathbb{N}$, $h = 1/N$, $z_0, \dots, z_N \in \mathbb{R}$ satisfying $z_N = z_0$ and $\sum_{i=0}^N z_i = 0$. Then*

$$h \sum_{i=0}^{N-1} z_i^2 \leq C_p h^{-1} \sum_{i=0}^{N-1} (z_{i+1} - z_i)^2,$$

where $C_p = h^2 / (4 \sin^2(h\pi)) \geq 1 / (4\pi^2)$. This constant is sharp.

These lemmas are stated in [23, Theorem 1]; for proofs see [15, p. 185] (Lemma 11) and [22, Theorem 1] (Lemma 12).

Lemma 13. *Let $a, b > 0$ and $x, y \geq 0$. Then*

$$(x^a - y^a)(x^b - y^b) \leq (x^{(a+b)/2} - y^{(a+b)/2})^2 \leq \frac{(a+b)^2}{4ab}(x^a - y^a)(x^b - y^b).$$

Proof. The second inequality is proven in [8, Lemma A.3]. For the proof of the first inequality, we divide it by y^{a+b} and set $z = x/y$. Then the inequality is equivalent to

$$(z^a - 1)(z^b - 1) \leq (z^{(a+b)/2} - 1)^2,$$

which after expansion can be equivalently written as $(z^{a/2} - z^{b/2})^2 \geq 0$, and this is true. \square

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INSTITUTE FOR ANALYSIS AND SCIENTIFIC COMPUTING, VIENNA UNIVERSITY OF TECHNOLOGY,
WIEDNER HAUPTSTRASSE 8-10, 1040 WIEN, AUSTRIA

E-mail address: `juengel@tuwien.ac.at`

INSTITUTE FOR ANALYSIS AND SCIENTIFIC COMPUTING, VIENNA UNIVERSITY OF TECHNOLOGY,
WIEDNER HAUPTSTRASSE 8-10, 1040 WIEN, AUSTRIA

E-mail address: `stefan.schuchnigg@tuwien.ac.at`